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On Itoh's finite amplitude stability theory for pipe flow

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In two recent papers Itoh has developed a finite amplitude stability theory which indicates that nonlinearity *increases* the damping rate of a small but finite amplitude disturbance to flow in a circular pipe when the disturbance is concentrated near the axis of the pipe. For such a *centre* mode, which is the only mode considered by Itoh, Davey & Nguyen found, in an earlier paper, the opposite result that nonlinearity *decreases* the damping rate. We examine the reasons for this discrepancy and we explain the subtle difference between Itoh's method and the method of Reynolds & Potter, which was used by Davey & Nguyen.

We suggest that for the centre mode of pipe flow neither Itoh's result nor Davey & Nguyen's result is a reliable guide to the true situation. However, for the wall mode of pipe flow, and especially for plane Couette flow, both methods give very similar results and we suggest that this similarity indicates that in these cases the damping rate is *decreased* by nonlinearity. For a particular problem we believe that it is only when the results of the two methods are very similar that either method is likely to be useful.

1. Introduction

It is well known that Poiseuille flow in a circular pipe is stable to infinitesimal disturbances, so that there is no neutral-stability curve for linearized theory. However, it is found experimentally that pipe flow usually becomes unstable when the Reynolds number based on the pipe radius exceeds a value of about 2000. It seems likely therefore that pipe flow may become unstable if a disturbance of small but finite amplitude exists in the flow whose amplitude just exceeds some critical value which we shall call the *equilibrium amplitude*. A disturbance whose amplitude is just larger than this equilibrium value will grow whereas one whose amplitude is just smaller will decay; if such an equilibrium state exists it is said to be unstable. In this paper we restrict attention to axisymmetric disturbances which, when they are infinitesimal and the Reynolds number is large, are concentrated either near the axis of the pipe, the *centre* modes, or near the wall of the pipe, the *wall* modes; see, for example, Gill (1965).

In an appendix to Davey & Nguyen (1971), Gill has given some simple physical arguments as to how the equilibrium amplitudes, if they exist, vary with the Reynolds number R for both the centre modes and the wall modes. For both modes he finds that the amplitudes vary as inverse powers of R. An initial attempt, using a nonlinear stability theory, to determine whether such equilibrium amplitudes exist, and if so their values, was made by Davey & Nguyen (1971) using the equilibrium amplitude method of false problems pioneered by Reynolds & Potter (1967). They found that equilibrium amplitudes do exist for both the least damped centre mode and for the least damped wall mode of linearized stability theory. Moreover the numerical values which they found depended upon R, for large values of R, exactly as predicted by Gill.

Recently Itoh (1977*a*, *b*) has used another nonlinear stability theory to investigate the same problem. He restricted his attention to the least damped centre mode and found, in contrast to Davey & Nguyen (1971), that an equilibrium amplitude did not exist. Whether such an equilibrium amplitude exists or not depends upon the sign of the imaginary part of the Landau constant. In all nonlinear stability theories the value of the Landau constant is given by the ratio of two integrals. These integrals depend upon the fundamental solution of the Orr–Sommerfeld equation and the associated adjoint function, the first harmonic of the fundamental and the distortion of the mean motion. Itoh's equations for these functions are exactly the same as those of Davey & Nguyen except for the harmonic equation, where one term is different. Nevertheless this single term makes a crucial difference to the value of the Landau constant for the centre mode, the reason for this being that the eigenvalues of the axisymmetric form of the Orr–Sommerfeld operator for the centre modes have a rather special distribution. This special distribution has been pointed out by Itoh (1977*b*) and we shall discuss its consequences.

In § 2 we explain the equilibrium amplitude method of Reynolds & Potter via a simple model problem which illustrates the key features of the method. We are also able to use the model problem to indicate how Itoh's method differs from that of Reynolds & Potter. This enables us to pinpoint the weaknesses of both methods and to explain why both methods should give very similar values for the Landau constant. In § 3 we present typical numerical results, using both methods, for three cases: (i) the least damped centre mode of pipe flow, (ii) the least damped wall mode of pipe flow and (iii) the least damped mode of plane Couette flow. In § 4 we discuss the results for these three cases and the conclusions which may be inferred from them.

2. The solution of a model problem by both methods

We suppose that the fluid is incompressible and has kinematic viscosity ν, that the pipe is infinitely long and of radius a, and that the externally applied mean pressure gradient is maintained at a constant value. The undisturbed flow is parabolic with speed U along the centre-line. We choose U and a as the characteristic speed and length with respect to which we make our quantities non-dimensional. Let x and r be the non-dimensional co-ordinates in the streamwise and radial directions respectively. We define a Reynolds number by

$$R = Ua/\nu. \tag{1}$$

We shall examine the stability of the flow to axially symmetric disturbances only, so that quantities will be invariant with respect to the azimuthal angle. This enables us to represent the velocity components via a stream function ϕ and we may eliminate the pressure by taking the curl of the Navier–Stokes equations.

We seek a marginally stable disturbance of small but finite amplitude A and frequency ω which is periodic in the distance x downstream with wavenumber α. Hence we define a phase function

$$\theta = \alpha x - \omega t \tag{2}$$

and seek a solution for ϕ of the form

$$\phi = \phi_0(r) + \phi_1(r)\exp(i\theta) + \tilde{\phi}_1(r)\exp(-i\theta) + \phi_2(r)\exp(2i\theta) + \dots, \tag{3}$$

where a tilde denotes the complex conjugate. The governing differential equation for ϕ for the pipe-flow problem is rather complicated and in order to explain the method of Reynolds & Potter (hereinafter whenever we refer to the method of Reynolds & Potter we shall mean their equilibrium amplitude method, see § 4 of their paper) it is sufficient here to consider the model problem whose governing differential equation is

$$\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}-R\left\{(1-r^2)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right\}\right]\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}\right]\phi = \phi^2. \qquad (4)$$

The linear operator on the left-hand side of (4) is just as for the pipe-flow problem but the simple term ϕ^2 on the right-hand side replaces the complicated nonlinear term.

In order to solve (4) we substitute the expansion (3) for ϕ into (4) and equate to zero the coefficient of each Fourier component. We shall need only the equations for ϕ_0, ϕ_1 and ϕ_2 and these are

$$(L_1+\omega M_1)\phi_1 = 2\phi_0\phi_1+2\phi_2\tilde{\phi}_1+\text{h.o.t.}, \qquad (5)$$

$$(L_2+2\omega M_2)\phi_2 = \phi_1^2+\text{h.o.t.}, \qquad (6)$$

$$L_0\phi_0 = 2\phi_1\tilde{\phi}_1+\text{h.o.t.}, \qquad (7)$$

where
$$L_n \equiv \left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-n^2\alpha^2-in\alpha R(1-r^2)\right]\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-n^2\alpha^2\right]$$

and
$$M_n \equiv iR\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-n^2\alpha^2\right].$$

In (5)–(7), h.o.t. refers to higher-order terms which, as we shall explain presently, will not be needed.

In order to solve (5)–(7) we recall that the real amplitude A of the disturbance is supposed to be small and so we can expand ϕ_0, ϕ_1, ϕ_2, ... as power series in A. Since ϕ_1 is of order A it follows from (6) that ϕ_2 will be of order A^2 and from (7) that ϕ_0 also will be of order A^2. Moreover the nth harmonic will be of order A^n. We also need to expand ω as a power series in A and, with some hindsight, we seek a solution of the form

$$\phi_1 = A\phi_{11}+A^3\phi_{13}+O(A^5), \qquad (8a)$$

$$\phi_2 = A^2\phi_{22}+O(A^4), \qquad (8b)$$

$$\phi_0 = A^2\phi_{02}+O(A^4), \qquad (8c)$$

$$\omega = \omega_0+A^2\omega_2+O(A^4). \qquad (8d)$$

The boundary conditions will be such as to imply that the solution of (4), and of (5)–(7), poses a nonlinear eigenvalue problem for ω as a function of A. For an arbitrary real value of A, ω will then in general be *complex*, but since we are seeking a solution which is marginally stable we need to determine the smallest value of A for which the imaginary part of ω is zero. Note that when A has been chosen such that ω is real then, provided that it converges, Reynolds & Potter's method yields an equilibrium state for *every* harmonic.

So we substitute (8) in (5)–(7) and equate coefficients of like powers of A. From the terms of order A in (5) we obtain the Orr–Sommerfeld equation

$$(L_1+\omega_0 M_1)\phi_{11} = 0. \qquad (9)$$

The solution of this equation yields the eigenfunction ϕ_{11} and the complex eigenvalue ω_0 of linear stability theory. The imaginary part ω_{0i} of ω_0 will be negative because there is no neutral-stability curve, so that infinitesimal disturbances are damped. From the terms of order A^2 in (6) we obtain what is usually called the harmonic equation,

$$(L_2+2\omega_0 M_2)\,\phi_{22} = \phi_{11}^2, \tag{10}$$

and from the terms of order A^2 in (7) we obtain the equation for the distortion of the mean motion,

$$L_0\phi_{02} = 2\,|\phi_{11}|^2. \tag{11}$$

Equations (10) and (11) are inhomogeneous two-point boundary-value problems determining ϕ_{22} and ϕ_{02} respectively after ϕ_{11} has been found from (9).

The key quantity which we wish to determine is the Landau constant ω_2 and we find this by writing down the terms of order A^3 in (5). The resulting equation has a solution only if a certain compatibility condition is satisfied which is obtained by multiplying the equation by the function adjoint to ϕ_{11} and integrating over the range of integration. This condition gives the value of ω_2 as the ratio of two integrals. The important point is that the values of these integrals, and hence the value of ω_2, are determined solely by the details of (9)–(11).

Because A is to be chosen such that ω is real it follows from (8 *d*) that we must have

$$0 = \omega_{0i}+A^2\omega_{2i}+O(A^4), \tag{12}$$

and so if the term of order A^4 in (12) is ignored then the amplitude of the disturbance is given by

$$A_e^2 = -\,\omega_{0i}/\omega_{2i}. \tag{13}$$

Thus, since ω_{0i} is negative, it is vital that ω_{2i} should be positive for an equilibrium amplitude A_e to exist.

The above is a description of the method of Reynolds & Potter as applied to problems without a neutral-stability curve. Fortunately we can easily use the above description of the model problem to explain the essence of how Itoh's method differs from the method of Reynolds & Potter. Itoh seeks a solution in the phase space of his amplitude functions along a line on which the rate of change with respect to time of the amplitude of the harmonic is zero. The analogy of this as regards the model problem discussed above, in which the amplitude of the harmonic is $A\exp(2\omega_i t)$ with A independent of time, is to set $\omega_i = 0$ directly in (6) *before* (6) is solved by means of the amplitude expansions (8). The result of doing this is that the equation for the harmonic function ϕ_{22}, instead of being (10), becomes

$$(L_2+2\omega_{0r} M_2)\,\phi_{22} = \phi_{11}^2, \tag{14}$$

the quantity $2\omega_0$ being changed to $2\omega_{0r}$. This is exactly the form of the harmonic equation used by Itoh and because (14) is different from (10) his method gives a different value for ω_2 as the integrals mentioned above are altered.

We thus see that Reynolds & Potter's method can be modified slightly to yield the equations which Itoh's method uses to calculate ω_2, simply by making use of (12), i.e. $\omega_i = 0$, at an earlier stage in the perturbation solution instead of just at the last step. Note that one could reason along similar lines that one might as well put $\omega_i = 0$ in (5) also before this equation is solved via the amplitude expansion (8), *but* if one does this then at order A, instead of the Orr–Sommerfeld equation (9), one obtains

$$(L_1 + \omega_{0r} M_1)\, \phi_{11} = 0, \tag{15}$$

and since ω_{0r} is *real* this equation has only the trivial solution $\phi_{11} = 0$!

The model problem mentioned above and the two methods of solution are reminiscent of a similar difficulty which arises in studies of strong–weak interactions in quantum field theory. Such problems require the solution of a nonlinear equation

$$L\psi = N\psi, \tag{16}$$

where L is a linear operator and N is a nonlinear operator. A common method to solve (16) is to use perturbation theory to solve instead

$$(H + \epsilon L)\, \psi = \epsilon(H + N)\, \psi, \tag{17}$$

where H is another, suitably chosen linear operator, and hence obtain a series solution for ψ. The required ψ is then found by setting $\epsilon = 1$. Here also the difficulty is that different answers can be obtained, when the series is truncated after a prescribed number of terms, if use is made of $\epsilon = 1$ somewhere before the last step.

Since Itoh's harmonic equation can be obtained from Reynolds & Potter's method by making use of the condition $\omega_i = 0$ before the last step, it follows that the difference between the values of the first Landau constant ω_2 given by the two methods is essentially due to a rearrangement of the terms in an infinite power series which is perhaps being used close to its radius of convergence. Both Reynolds & Potter's method and Itoh's method set $d(\text{mean-motion distortion})/dt \equiv 0$ and hence they both include higher-order amplitude terms at a lower order and this is equivalent to another series rearrangement. The above identity has to be imposed by *any* Landau-type method when there is no neutral-stability curve as otherwise a compatibility condition cannot be obtained and so the Landau constants cannot be clearly defined.

Since the difference between the two methods amounts to a rearrangement of the terms of an infinite series, the crucial point is whether or not the two series are being used inside their respective radii of convergence. Only when we are well inside the radii of convergence of *both* series will the two-term truncation results obtained by the two methods be very similar. If the results are very different then at least one series is probably being used outside its radius of convergence. We suggest that if only two terms are used then it is difficult to see which method is preferable when they do not agree. The only way to resolve this question would be to take the calculations to higher order, then recast the series to extend the radii of convergence. We predict that both methods would give identical results if this could be accomplished.

We feel therefore that only when Reynolds & Potter's method and Itoh's method give very similar results for the Landau constant ω_2 is either result likely to be a reliable guide to the true situation. With this last thought in mind we present in § 3 numerical results for three cases, obtained by using both forms of the equation for the harmonic.

3. Numerical results for pipe flow and for plane Couette flow

As we mentioned in § 2, the calculation of the Landau constant ω_2 is quite straightforward: the only decision which needs to be taken is whether to include or omit the term involving $2\omega_{0i}$ from the harmonic equation. In order to see how the Landau

λ	ω_2	λ	ω_2
0	$-2{\cdot}4+23{\cdot}2i$	0·6	$+34{\cdot}9-18{\cdot}5i$
0·1	$-6{\cdot}9+\ 0{\cdot}5i$	0·7	$+37{\cdot}7-16{\cdot}8i$
0·2	$+3{\cdot}6-15{\cdot}9i$	0·8	$+39{\cdot}6-15{\cdot}4i$
0·3	$+15{\cdot}9-21{\cdot}4i$	0·9	$+41{\cdot}1-14{\cdot}2i$
0·4	$+24{\cdot}9-21{\cdot}6i$	1	$+42{\cdot}3-13{\cdot}1i$
0·5	$+30{\cdot}9-20{\cdot}2i$		

TABLE 1. The variation of ω_2 with λ for the least damped centre mode of pipe flow when $\alpha = 6{\cdot}2$ and $R = 500$. $\lambda = 0$ corresponds to Reynolds & Potter's method and $\lambda = 1$ to Itoh's method.

constant varied between the two methods we did a series of calculations with the term involving $2\omega_{0i}$ multiplied by $1-\lambda$ with $\lambda = 0(0{\cdot}1)1$. Hence the case $\lambda = 0$ corresponds to Reynolds & Potter's method and the case $\lambda = 1$ corresponds to Itoh's method.

Case (i): the centre mode for pipe flow

Davey & Nguyen considered the temporal stability problem, so that their wavenumber α was real, and they found that when R was large (see figure 6 of their paper) the most dangerous wavenumber, from the point of view of its being the one most likely to lead to transition, was approximately $0{\cdot}77R^{\frac{1}{3}}$ for the least damped centre mode. In accordance with this result we did calculations for the temporal stability problem with $\alpha = 6{\cdot}2$ and $R = 500$, for which values

$$\omega_0 = 5{\cdot}8850-0{\cdot}3918i. \tag{18}$$

We also did calculations for other values of α and R but the results which we present in table 1 are typical of those which we obtained for the other values.

Since the rows of table 1 corresponding to $\lambda = 0$ and $\lambda = 1$ are, apart from a scale factor, the same as the numerical values quoted by Itoh (1977*b*, p. 477), we are in complete agreement with his numerical work. Note that the results for $\lambda = 0$ and $\lambda = 1$ are very different, even to the extent that the imaginary part of ω_2 changes sign. This sign change means [see (13)] that Davey & Nguyen obtained an equilibrium amplitude whereas Itoh claimed that the nonlinearity has a stabilizing effect. This dichotomy has been well discussed by Itoh (1977*b*, p. 477), who correctly pointed out that the eigenvalues of the centre-mode problem have a rather special distribution in that $2\omega_0$ is very close to an eigenvalue of the left-hand-side harmonic operator $L_2+2\omega_0 M_2$, i.e. of the Orr–Sommerfeld operator with α replaced by 2α. The solution of the harmonic equation with the $2\omega_{0i}$ term multiplied by $1-\lambda$ is therefore very sensitive to changes in λ as table 1 indicates.

Case (ii): the wall mode for pipe flow

For the least damped wall mode Davey & Nguyen found that when R was large the wavenumber of the most dangerous disturbance was approximately $0{\cdot}145R^{\frac{1}{2}}$ (see figure 7 of their paper). In accordance with this result we did calculations for this case with $\alpha = 5{\cdot}8$ and $R = 1600$, for which values

$$\omega_0 = 1{\cdot}5847-0{\cdot}5395i. \tag{19}$$

Again we did calculations for many other values of α and R but the results which we present in table 2 are typical of all the other calculations.

λ	$10^{-3}\,\omega_2$	λ	$10^{-3}\,\omega_2$
0	$538+406i$	0·6	$1004+416i$
0·1	$632+441i$	0·7	$1043+393i$
0·2	$726+460i$	0·8	$1073+371i$
0·3	$813+464i$	0·9	$1096+351i$
0·4	$890+455i$	1	$1114+333i$
0·5	$953+438i$		

TABLE 2. The variation of ω_2 with λ for the least damped wall mode of pipe flow when $\alpha = 5{\cdot}8$ and $R = 1600$. $\lambda = 0$ corresponds to Reynolds & Potter's method and $\lambda = 1$ to Itoh's method.

In contrast to the centre-mode case we see that for the wall mode at least the imaginary part of ω_2 is relatively unchanged whichever harmonic equation is used, as can be seen from the rows of table 2 corresponding to $\lambda = 0$ and $\lambda = 1$. In particular, note that ω_{2i} is positive for both $\lambda = 0$ and $\lambda = 1$, so that (13) yields an equilibrium amplitude for Reynolds & Potter's method, while Itoh's method claims that the nonlinearity has a destabilizing effect. The real part of ω_2 does vary rather more between $\lambda = 0$ and $\lambda = 1$ but this is not too important as the physical significance of the real part of ω_2 is just the extent to which nonlinearity alters the phase speed of the disturbance. (The large numerical values for ω_2 are no cause for concern; this is due solely to the fact that we normalized the Orr–Sommerfeld eigenfunction by setting it equal to $r^2+O(r^4)$ for r small; this is appropriate for the centre mode but not for the wall mode.)

Case (iii): plane Couette flow

Another flow which has no neutral-stability curve is plane Couette flow. For this flow, when R is large, the disturbance will be concentrated near one of the boundaries, and as Davey & Nguyen have explained, the stability problem is similar to the wall-mode problem for pipe flow. When R is large the wavenumber α of the most dangerous disturbance is approximately $0{\cdot}19R^{\frac{1}{2}}$ (see figure 8 of their paper).† Again we have done many calculations for various values of α and R but those which we present in table 3 for $\alpha = 4{\cdot}6$ and $R = 625$, when

$$\omega_0 = 7{\cdot}3127-0{\cdot}7109i, \tag{20}$$

are typical of all the results which we obtained.

For this case the imaginary part of ω_2 is almost the same for $\lambda = 0$ as for $\lambda = 1$. Moreover for both values of λ it is negative, so that, from (13), Reynolds & Potter's method yields an equilibrium amplitude and Itoh's method claims that the nonlinearity has a destabilizing effect. There is again, as for the wall mode of pipe flow, more variation in the real part of ω_2. The result (20) is for a disturbance which is concentrated near the boundary which moves with non-dimensional speed 2 and the phase speed of linear stability theory is 1·59. Owing to nonlinear effects the phase speed at the equilibrium amplitude becomes 1·48 when $\lambda = 0$ and 1·28 when $\lambda = 1$. Thus Itoh's method produces a considerably larger change in the phase speed; this is also true for the previous case.

† As in Davey & Nguyen we suppose that the bounding planes are a distance h apart and that one plane is stationary while the other moves with speed $2U$. We define the Reynolds number by $R = Uh/\nu$ and the wavelength by $2\pi h/\alpha$.

λ	ω_2	λ	ω_2
0	$-5{\cdot}44+8{\cdot}01i$	0·6	$-13{\cdot}64+9{\cdot}10i$
0·1	$-7{\cdot}11+8{\cdot}89i$	0·7	$-14{\cdot}31+8{\cdot}75i$
0·2	$-8{\cdot}77+9{\cdot}44i$	0·8	$-14{\cdot}83+8{\cdot}40i$
0·3	$-10{\cdot}31+9{\cdot}66i$	0·9	$-15{\cdot}23+8{\cdot}10i$
0·4	$-11{\cdot}65+9{\cdot}62i$	1	$-15{\cdot}52+7{\cdot}82i$
0·5	$-12{\cdot}75+9{\cdot}41i$		

TABLE 3. The variation of ω_2 with λ for the least damped mode of plane Couette flow when $\alpha = 4{\cdot}6$ and $R = 625$. $\lambda = 0$ corresponds to Reynolds & Potter's method and $\lambda = 1$ to Itoh's method.

We should perhaps mention that in each of the three tables given above the values of ω_2 will be rescaled by a positive real factor if the associated Orr–Sommerfeld eigenfunction is renormalized. Since our main interest is to compare results for $\lambda = 0$ and $\lambda = 1$ the particular normalization used is immaterial. We now discuss the implications of the above numerical results.

4. Conclusions

In Reynolds & Potter's method the amplitude of the nth harmonic is supposed to be proportional to $A^n \exp(in\omega t)$. Then for a given real value of A, $\omega = \omega(A)$ is determined by a nonlinear eigenvalue problem and the smallest value of A is sought for which the corresponding value of ω is real, so that the amplitude of *every* harmonic will be in equilibrium. This search for a real value of ω is made along the line $dA/dt = 0$ in the phase space $[A, \omega]$. In Itoh's method, if a_n denotes the amplitude of his nth harmonic, a solution is sought in the phase space $[a_n]$ along the line $da_n/dt = 0$, $n \neq 1$, and he should then have looked to see if his expansion parameter ϵ could be chosen such that da_1/dt can also be zero. Thus there is a close similarity between the two methods.

Moreover we demonstrated in § 2 that this similarity can be seen in a different way in that a slight adjustment of the method of Reynolds & Potter yields the key equations which Itoh used to determine his Landau constant. We also pointed out that this adjustment is equivalent to a rearrangement of terms in an infinite series which may be being used close to its radius of convergence. If both methods were used to calculate the second Landau constant, i.e. the coefficient ω_4 of A^4 in (8d), then the different evaluations of ω_4 would hopefully counterbalance the different evaluations of ω_2, so that one would hope that when *three* terms are used in the expansion of ω_i to evaluate A_e^2 both methods will give closer answers than when only two terms are used.†

We believe that when only the first Landau constant ω_2 is calculated then, because the difference in the values of ω_2 obtained by the two methods is essentially due to a rearrangement of the terms in an infinite series, only when these values are very similar are they likely to be a reliable guide to the truth. In view of this belief and the numerical results presented in § 3 we suggest, therefore, that *neither* method resolves the pipe-flow centre-mode problem, that the pipe-flow wall-mode problem is likely to have an equilibrium amplitude and that the problem of plane Couette flow almost certainly

† An alternative approach (Herbert 1977) which may well be more rewarding is to solve numerically equations for the Fourier components such as (5)–(7), suitably truncated at a fairly high order, directly without using an amplitude expansion such as (8).

has an equilibrium amplitude. The pipe-flow centre-mode problem is particularly difficult because the special distribution of the eigenvalues of the Orr–Sommerfeld operator for this case, as mentioned in § 3, restricts the radius of convergence of (8d) so severely that however many Landau constants are calculated neither method is likely to be of any use.

For problems without a neutral-stability curve the principal weakness of both Reynolds & Potter's method and Itoh's method is that they attempt to solve a fully *nonlinear* problem by an expansion procedure in which the cross-space dependence of the leading term is governed by the Orr–Sommerfeld operator, an operator associated with *linear* stability theory. The cross-space dependence of the *exact* solution of the nonlinear problem may not be close to that for the least damped eigenfunction of the Orr–Sommerfeld operator.

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